Velocity and acceleration
GENERAL METHOD

$$f_1 = f_1(w_1, \ldots, w_k, q_1, \ldots, q_n, x_1, \ldots, x_m) = 0$$
$$f_2 = f_2(w_1, \ldots, w_k, q_1, \ldots, q_n, x_1, \ldots, x_m) = 0$$
$$\ldots$$
$$f_m = f_m(w_1, \ldots, w_k, q_1, \ldots, q_n, x_1, \ldots, x_m) = 0$$

w – vector of links' dimensions,
q – vector of known independent variables (drivers' position),
x – vector of unknown dependent variables (links' position)

$$\mathbf{f}(\mathbf{w}, \mathbf{q}, \mathbf{x}) = 0$$

$$\mathbf{q} = \mathbf{q}(t) \qquad \mathbf{x} = \mathbf{x}(t)$$

$$f = 0 \rightarrow \frac{df}{dt} = 0 \rightarrow \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial q}\dot{q} = 0$$

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \cdots & & & \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_m} \end{bmatrix} = \mathbf{A}$$

$$\dot{x} = [\dot{x}_1 \ldots . \dot{x}_m]^T$$

Dependent velocities

$$-\frac{\partial \mathbf{f}}{\partial \mathbf{q}} = -\begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} & \cdots & \frac{\partial f_1}{\partial q_n} \\ \cdots & & & \\ \frac{\partial f_m}{\partial q_1} & \frac{\partial f_m}{\partial q_2} & \cdots & \frac{\partial f_m}{\partial q_n} \end{bmatrix} = \mathbf{B}$$

$$\dot{q} = [\dot{q}_1 \ldots . \dot{q}_n]^T$$

Independent velocities, drivers

$$A\dot{x} = B\dot{q}$$

$$A^{-1}A = 1$$

$$\dot{x} = A^{-1}B\dot{q}$$

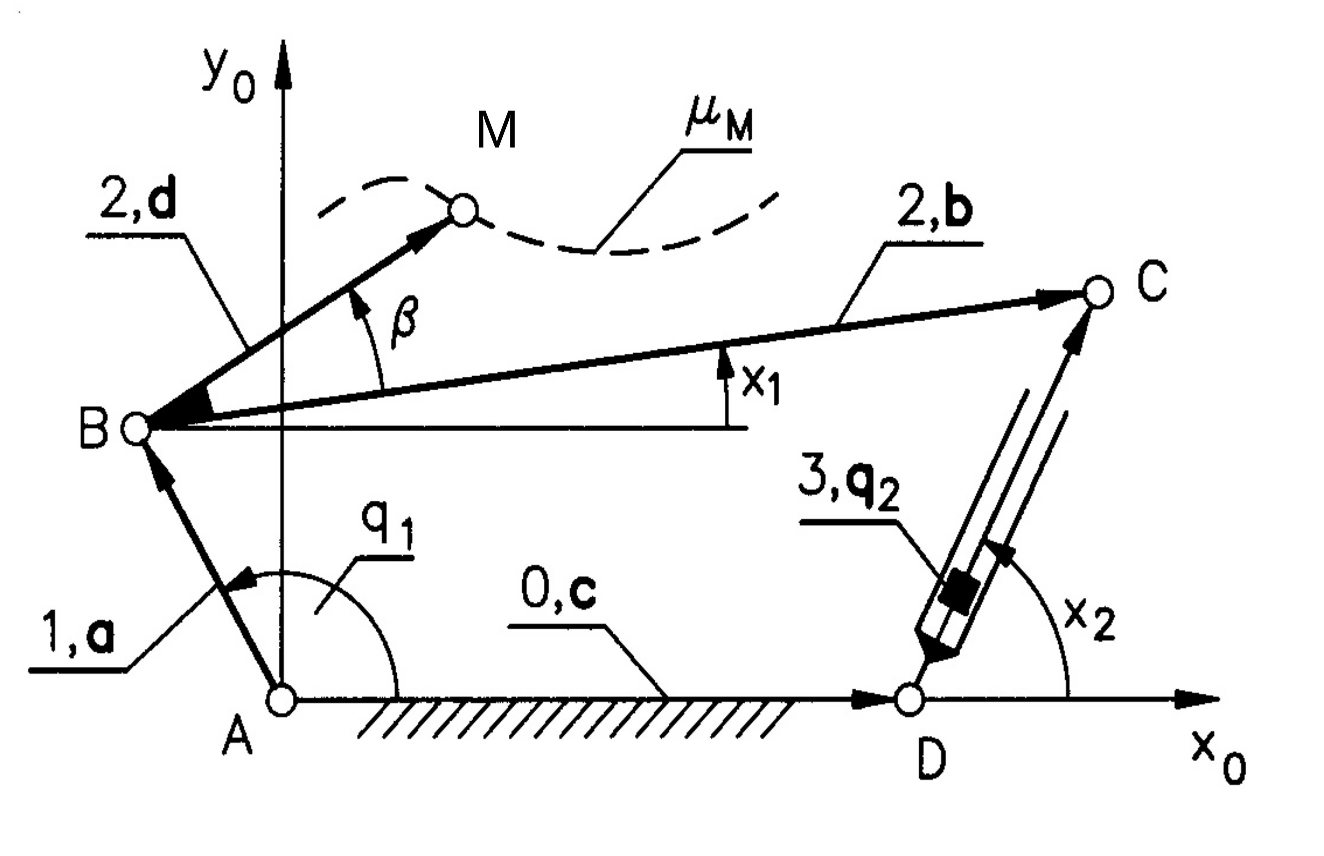
$$\ddot{x} = [\ddot{x}_1 \ldots . \ddot{x}_m]^T$$

$$\ddot{q} = [\ddot{q}_1 \ldots . \ddot{q}_n]^T$$

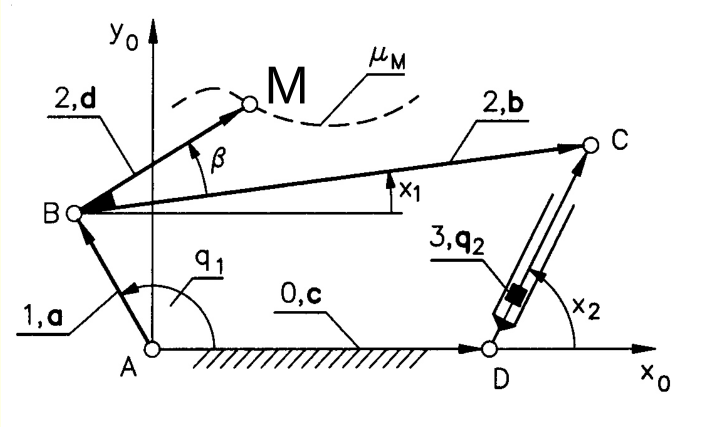
$$A\dot{x} = B\dot{q} \quad \Longrightarrow \quad A\ddot{x} + \dot{A}\dot{x} = B\ddot{q} + \dot{B}\dot{q}$$

$$\ddot{x} = A^{-1}\left(-\dot{A}\dot{x} + \dot{B}\dot{q} + B\ddot{q}\right)$$

Example of 2 DOF manipulator



$$q_1, q_2 \rightarrow x_M = \qquad y_M =$$



$$\mathbf{a}+\mathbf{b}-\mathbf{c}-\mathbf{q}_2=\mathbf{0} \quad \longrightarrow \quad \begin{aligned} a\cos q_1+b\cos x_1-c-q_2\cos x_2&=0\\ a\sin q_1+b\sin x_1-q_2\sin x_2&=0\end{aligned}$$

$$\mathbf{f}=\begin{bmatrix} f_1\\ f_2\end{bmatrix}=\begin{bmatrix} a\cos q_1+b\cos x_1-c-q_2\cos x_2\\ a\sin q_1+b\sin x_1-q_2\sin x_2\end{bmatrix}=0$$

q_1, q_2 – independent (drivers),

x_1, x_2 – unknowns,

$$\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} a\cos q_1 + b\cos x_1 - c - q_2\cos x_2 \\ a\sin q_1 + b\sin x_1 - q_2\sin x_2 \end{bmatrix} = 0$$

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \cdots & & & \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_m} \end{bmatrix} = \mathbf{A}$$

$$\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} -b\sin x_1 & q_2\sin x_2 \\ b\cos x_1 & -q_2\cos x_2 \end{bmatrix}$$

$$\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} a\cos q_1 + b\cos x_1 - c - q_2 \cos x_2 \\ a\sin q_1 + b\sin x_1 - q_2 \sin x_2 \end{bmatrix} = 0$$

$$-\frac{\partial \mathbf{f}}{\partial \mathbf{q}} = -\begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} & \cdots & \frac{\partial f_1}{\partial q_n} \\ \cdots & & & \\ \frac{\partial f_m}{\partial q_1} & \frac{\partial f_m}{\partial q_2} & \cdots & \frac{\partial f_m}{\partial q_n} \end{bmatrix} = \mathbf{B}$$

$$\mathbf{B} = -\frac{\partial \mathbf{f}}{\partial \mathbf{q}} = -\begin{bmatrix} -a\sin q_1 & -\cos x_2 \\ a\cos q_1 & -\sin x_2 \end{bmatrix}$$

$$\dot{\boldsymbol{x}} = \boldsymbol{A}^{-1}\boldsymbol{B}\dot{\boldsymbol{q}}$$

$$\dot{x} = [\dot{x}_1 \ \dot{x}_2]^T \qquad \dot{q} = [\dot{q}_1 \ \dot{q}_2]^T$$

Velocity equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -b \sin x_1 & q_2 \sin x_2 \\ b \cos x_1 & -q_2 \cos x_2 \end{bmatrix}^{-1} \begin{bmatrix} a \sin q_1 & \cos x_2 \\ -a \cos q_1 & \sin x_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

Velocity equation

$$\dot{\boldsymbol{x}} = \boldsymbol{A}^{-1}\boldsymbol{B}\dot{\boldsymbol{q}}$$

Acceleration equation

$$\ddot{x} = A^{-1}\left(-\dot{A}\dot{x} + \dot{B}\dot{q} + B\ddot{q}\right)$$

$$\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} -b\sin x_1 & q_2 \sin x_2 \\ b\cos x_1 & -q_2 \cos x_2 \end{bmatrix}$$

$$\dot{A} = \frac{d}{dt}A = \begin{bmatrix} -\dot{x}_1 b\cos x_1 & \dot{q}_2 \sin x_2 - \dot{x}_2 q_2 \cos x_2 \\ -\dot{x}_1 b\sin x_1 & \dot{q}_2 \cos x_2 + \dot{x}_2 q_2 \sin x_2 \end{bmatrix}$$

$$\mathbf{B} = -\frac{\partial \mathbf{f}}{\partial \mathbf{q}} = -\begin{bmatrix} -a\sin q_1 & -\cos x_2 \\ a\cos q_1 & -\sin x_2 \end{bmatrix}$$

$$\dot{B} = \frac{d}{dt}B = \begin{bmatrix} \dot{q_1} a\cos q_1 & -\dot{x_2}\sin x_2 \\ \dot{q_1} a\sin q_1 & \dot{x_2}\cos x_2 \end{bmatrix}$$

$$\ddot{x} = A^{-1}(-\dot{A}\dot{x} + \dot{B}\dot{q} + B\ddot{q})$$

$$\ddot{x} = [\ddot{x}_1, \ddot{x}_2]^T \qquad \ddot{q} = [\ddot{q}_1, \ddot{q}_2]^T$$

Acceleration equation

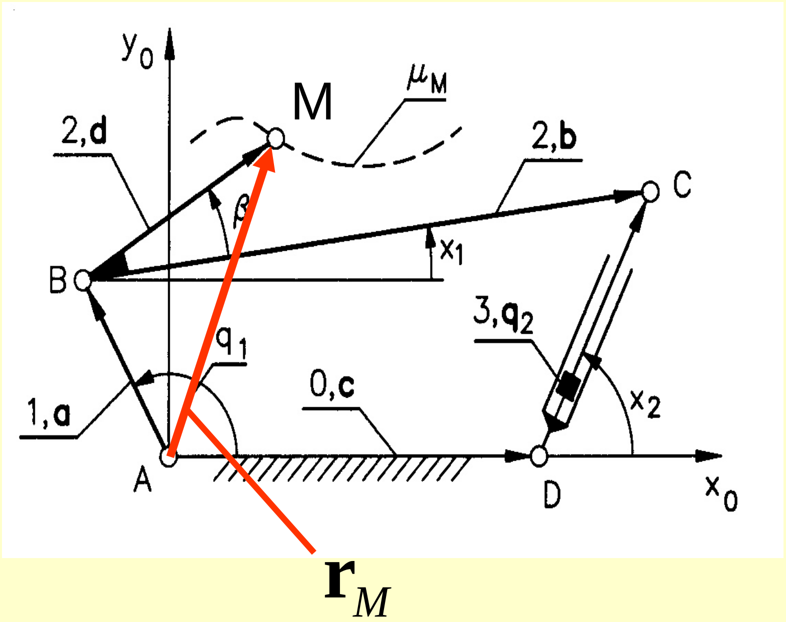
$$\mathbf{B} = -\frac{\partial \mathbf{f}}{\partial \mathbf{q}} = -\begin{bmatrix} -a\sin q_1 & -\cos x_2 \\ a\cos q_1 & -\sin x_2 \end{bmatrix}$$

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} =$$

$$= \begin{bmatrix} -b\sin x_1 & q_2 \sin x_2 \\ b\cos x_1 & -q_2\cos x_2 \end{bmatrix}^{-1} \left\{ -\begin{bmatrix} -\dot{x}_1 b\cos x_1 & \dot{q}_2 \sin x_2 - \dot{x}_2 q_2 \cos x_2 \\ -\dot{x}_1 b \sin x_1 & \dot{q}_2 \cos x_2 + \dot{x}_2 q_2 \sin x_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \right.$$

$$\left. + \begin{bmatrix} \dot{q}_1 a\cos q_1 & -\dot{x}_2 \sin x_2 \\ \dot{q}_1 a \sin q_1 & \dot{x}_2 \cos x_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} a\sin q_1 & -\cos x_2 \\ a\cos q_1 & -\sin x_2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} \right\}$$

$$\mathbf{r}_M = \mathbf{a} + \mathbf{d}$$



$$\begin{bmatrix} x_M \\ y_M \end{bmatrix} = \begin{bmatrix} a\cos q_1 + d\cos(x_1 + \beta) \\ a\sin q_1 + d\sin(x_1 + \beta) \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_M \\ \dot{y}_M \end{bmatrix} = \begin{bmatrix} -a\dot{q}_1 \sin q_1 - d\dot{x}_1 \sin(x_1 + \beta) \\ a\dot{q}_1 \cos q_1 + d\dot{x}_1 \cos(x_1 + \beta) \end{bmatrix}$$

$$\begin{bmatrix} \ddot{x}_M \\ \ddot{y}_M \end{bmatrix} = \begin{bmatrix} -a\ddot{q}_1 \sin q_1 - a\dot{q}_1^2 \cos q_1 - d\ddot{x}_1 \sin(x_1 + \beta) - d\dot{x}_1^2 \cos(x_1 + \beta) \\ a\ddot{q}_1 \cos q_1 - a\dot{q}_1^2 \sin q_1 + d\ddot{x}_1 \cos(x_1 + \beta) - d\dot{x}_1^2 \sin(x_1 + \beta) \end{bmatrix}$$

M A N I P U L A T O R S

(robot mechanisms)

SERIAL and PARALLEL

P L A N A R MANIPULATORS

APPLICATIONS of ROBOTS:

Operation in a danger zone:

- •RADIATION
- •EXPLOSION (POLICE, ARMY)
- •HIGH PRESSURE, HIGH TEMPERATURE

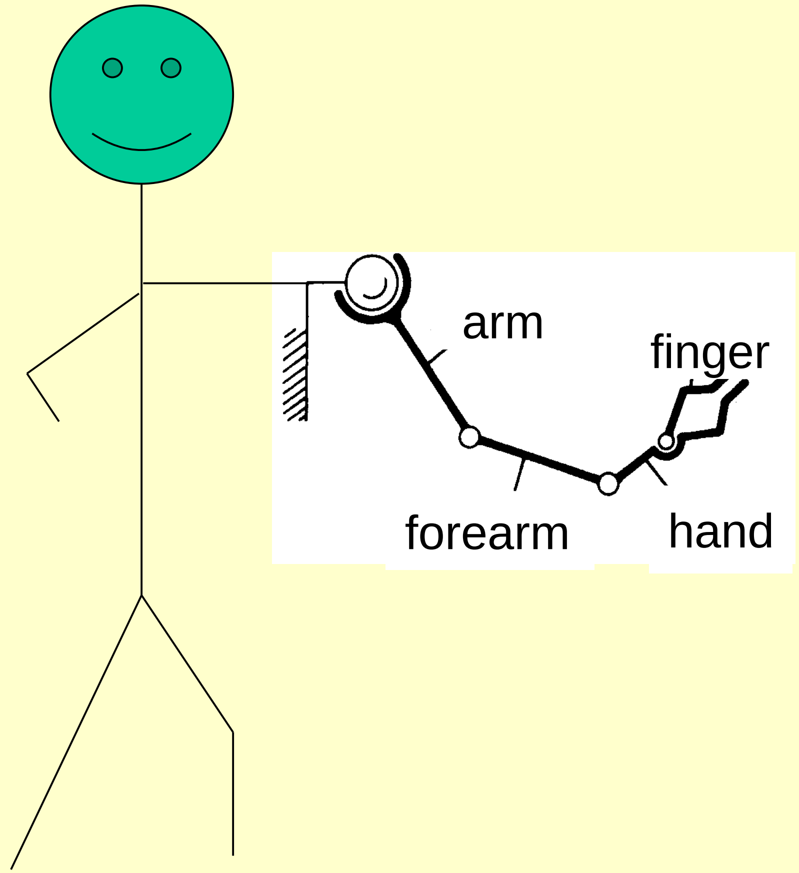
Manufacturing

- •ASSAMBLY, WELDING, MACHINING, etc

Health care

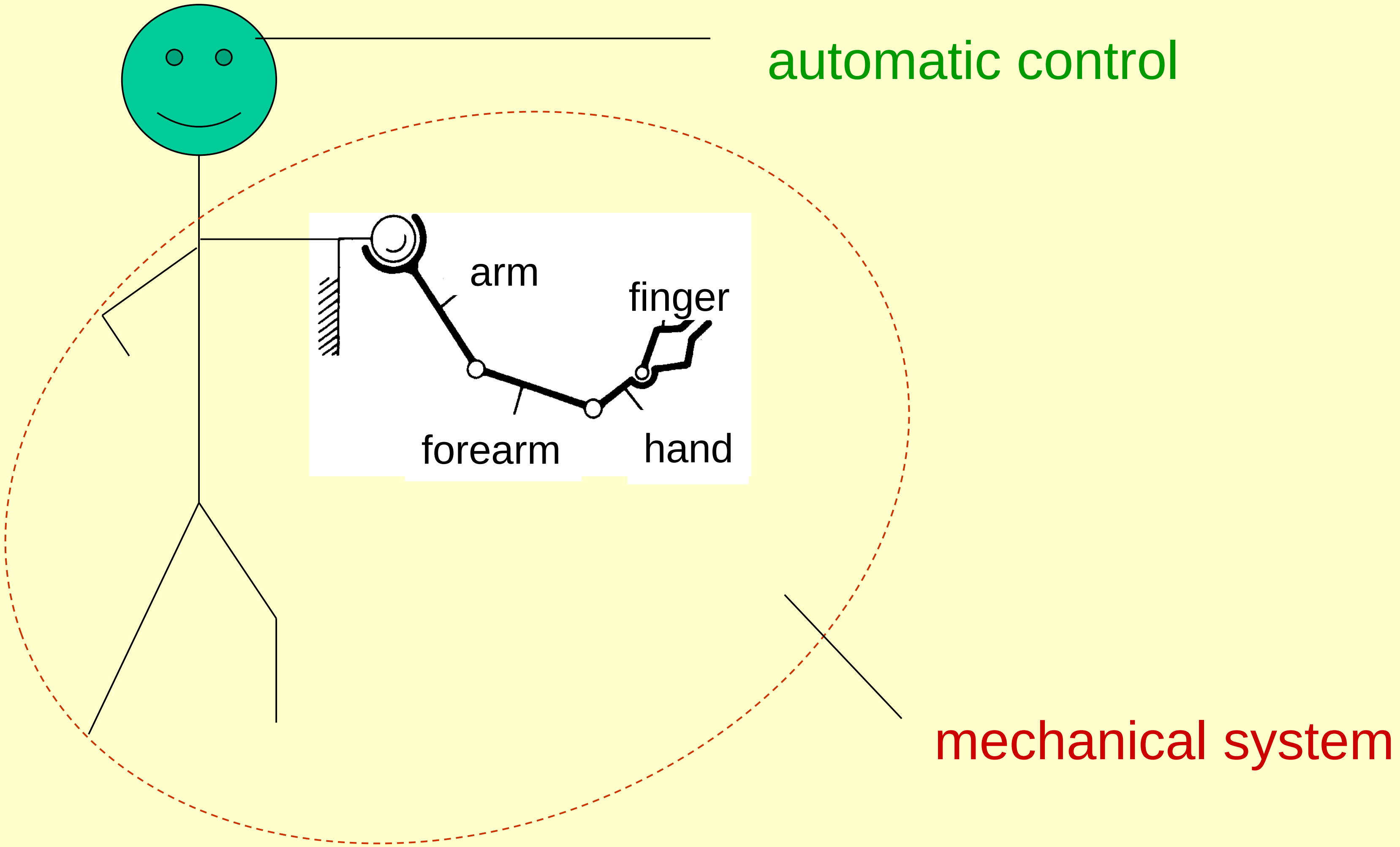
- •REHABILITATION
- •OPERATIONS

and many others ...



END-EFFECTOR
Device attached to the robot arm by which objects can be grasped or acted upon.

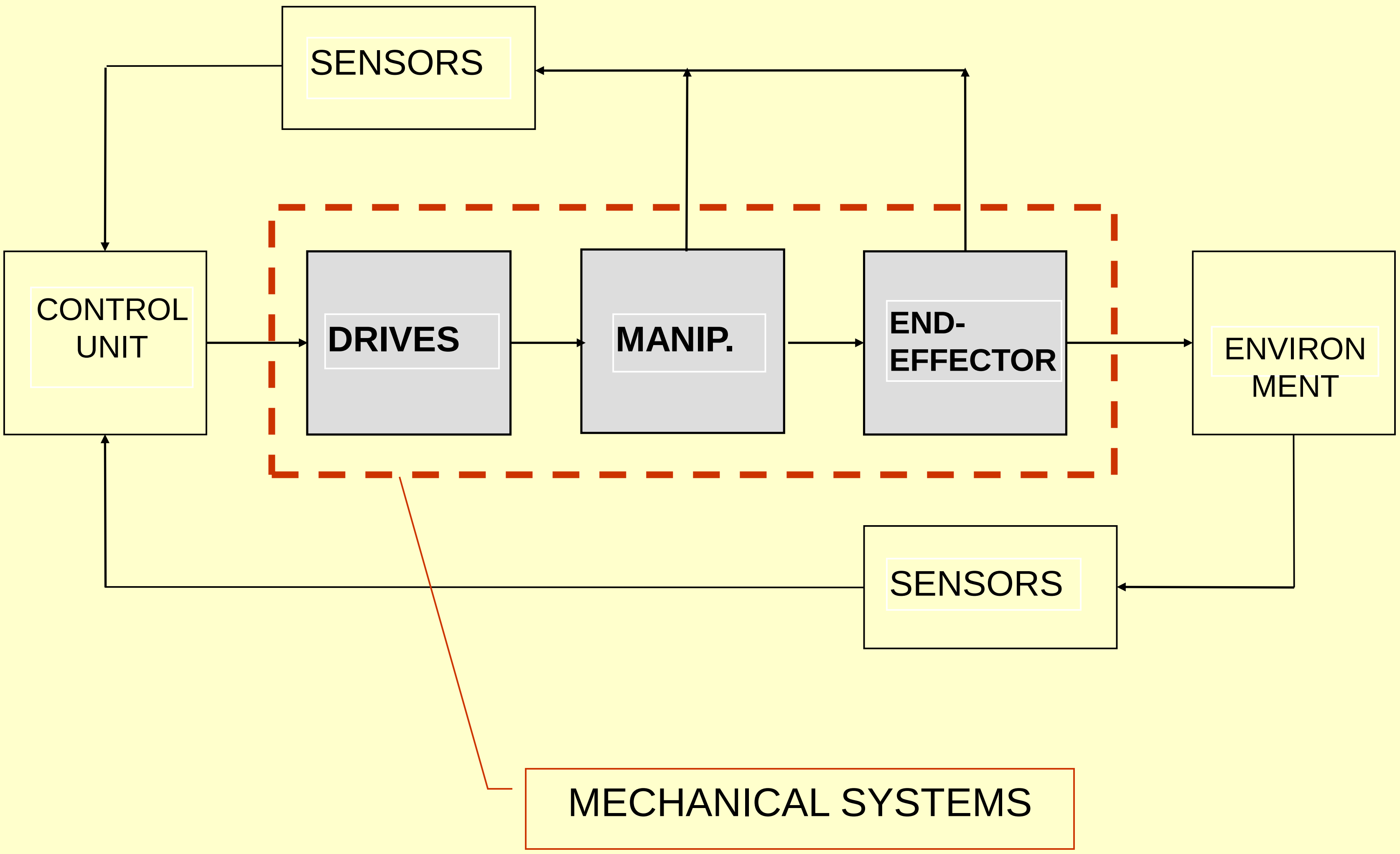
MANIPULATOR
Mechanical device for gripping and the controlled movement of objects (mechanism having many degrees of freedom)



ROBOT

Mechanical system under automatic control that performs operations such as handling and locomotion

Modern robot block diagram



PARALLEL MANIPULATORS

Manipulator that controls the motion of its end effector by means of at least two kinematic chains going from the end-effector towards the frame.

Planar parallel robot – 2 dof

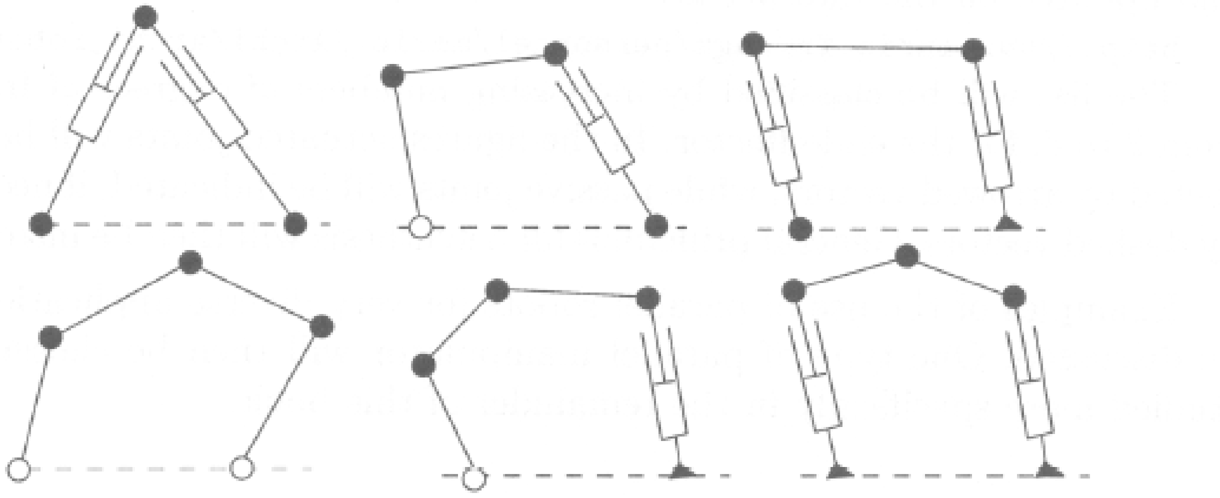
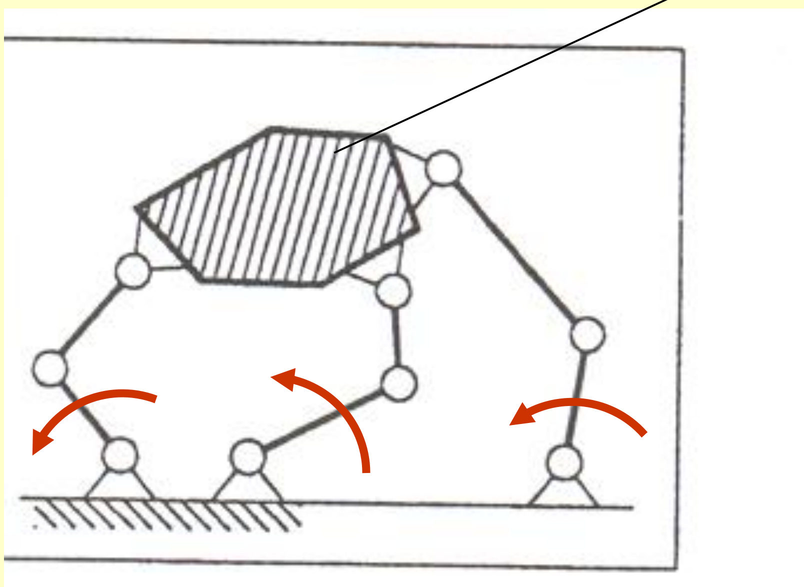


Figure 2.1. Two degrees of freedom planar robots, according to McCloy. White circles represent active revolute joints, black circles passive ones.

Parallel manipulators (2D)

end-effector (platform)

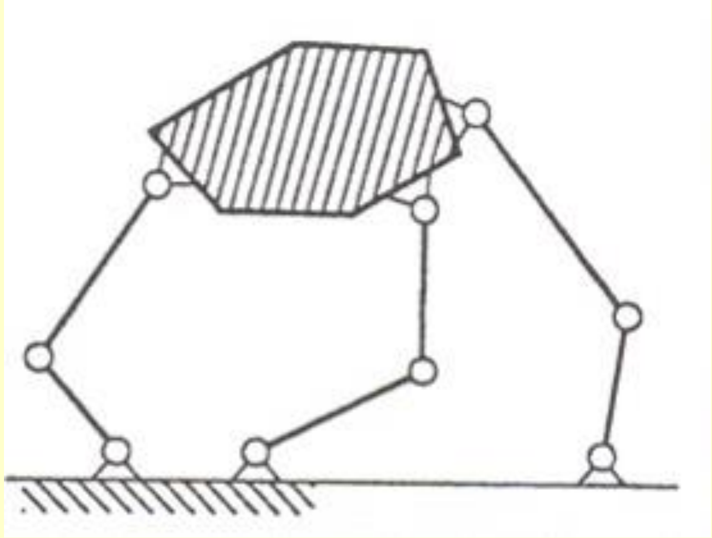


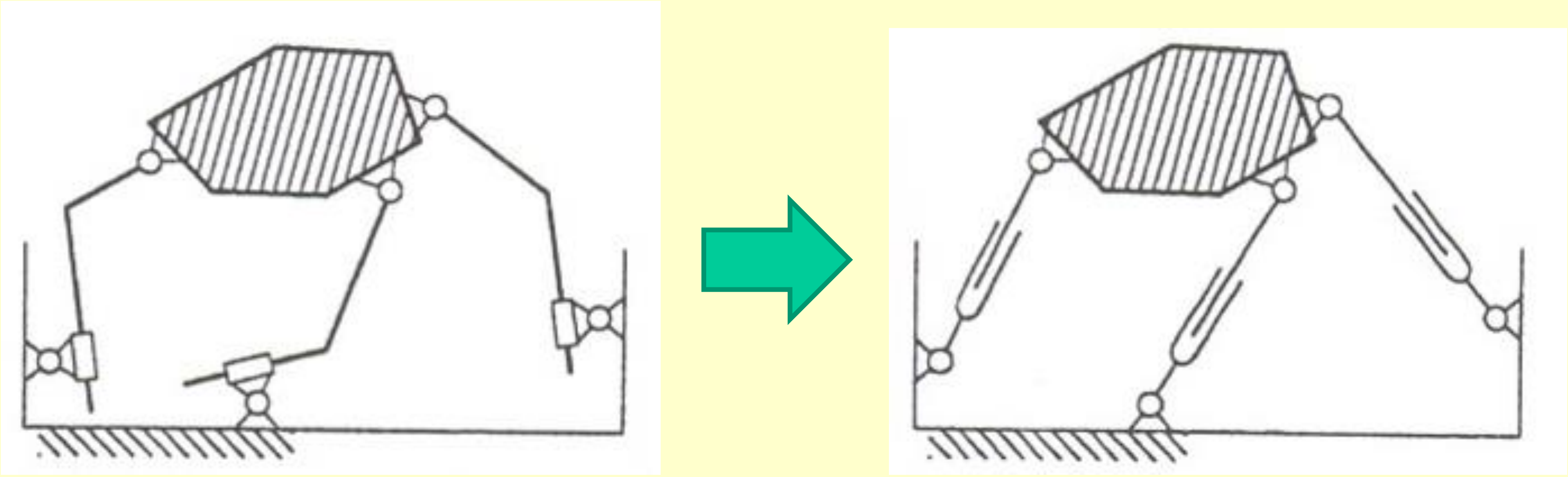
DOF = 3

circles are 1-st class joints:

R and/or T

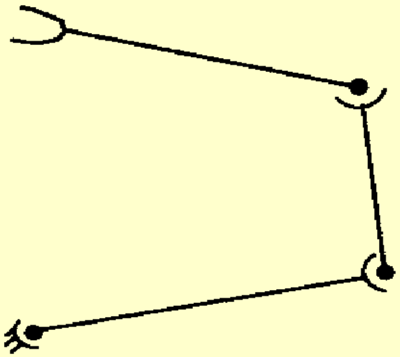
3 RRR

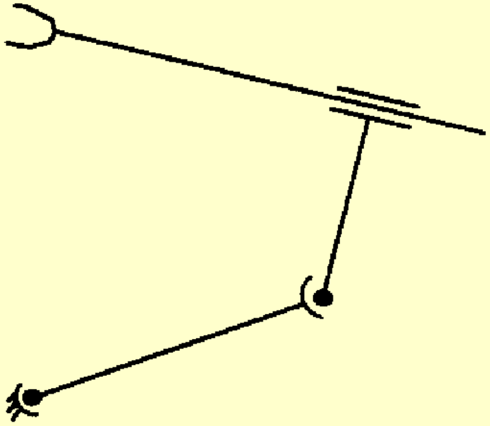




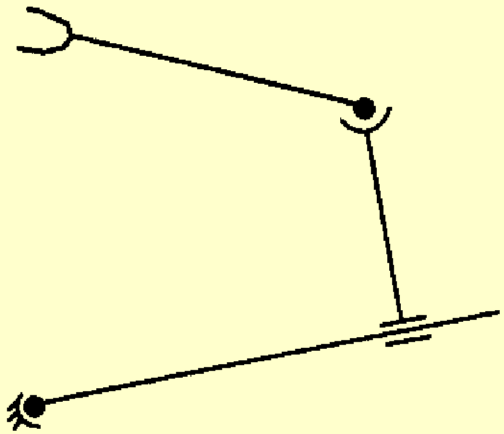
Planar serial manipulators DOF = 3

RRR

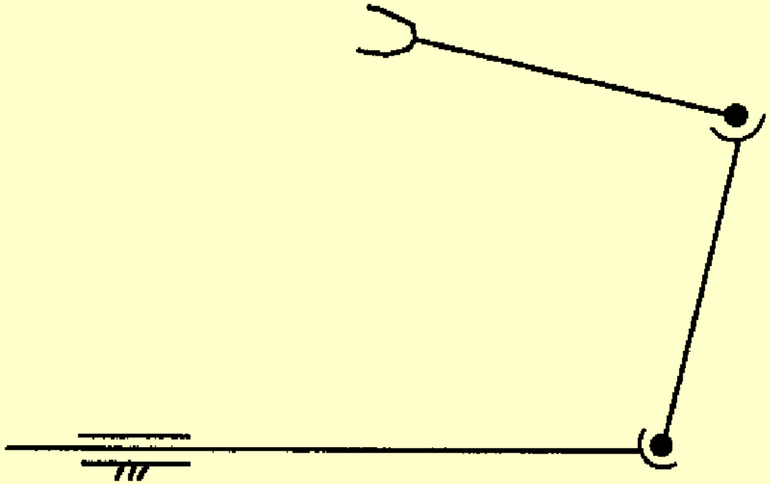




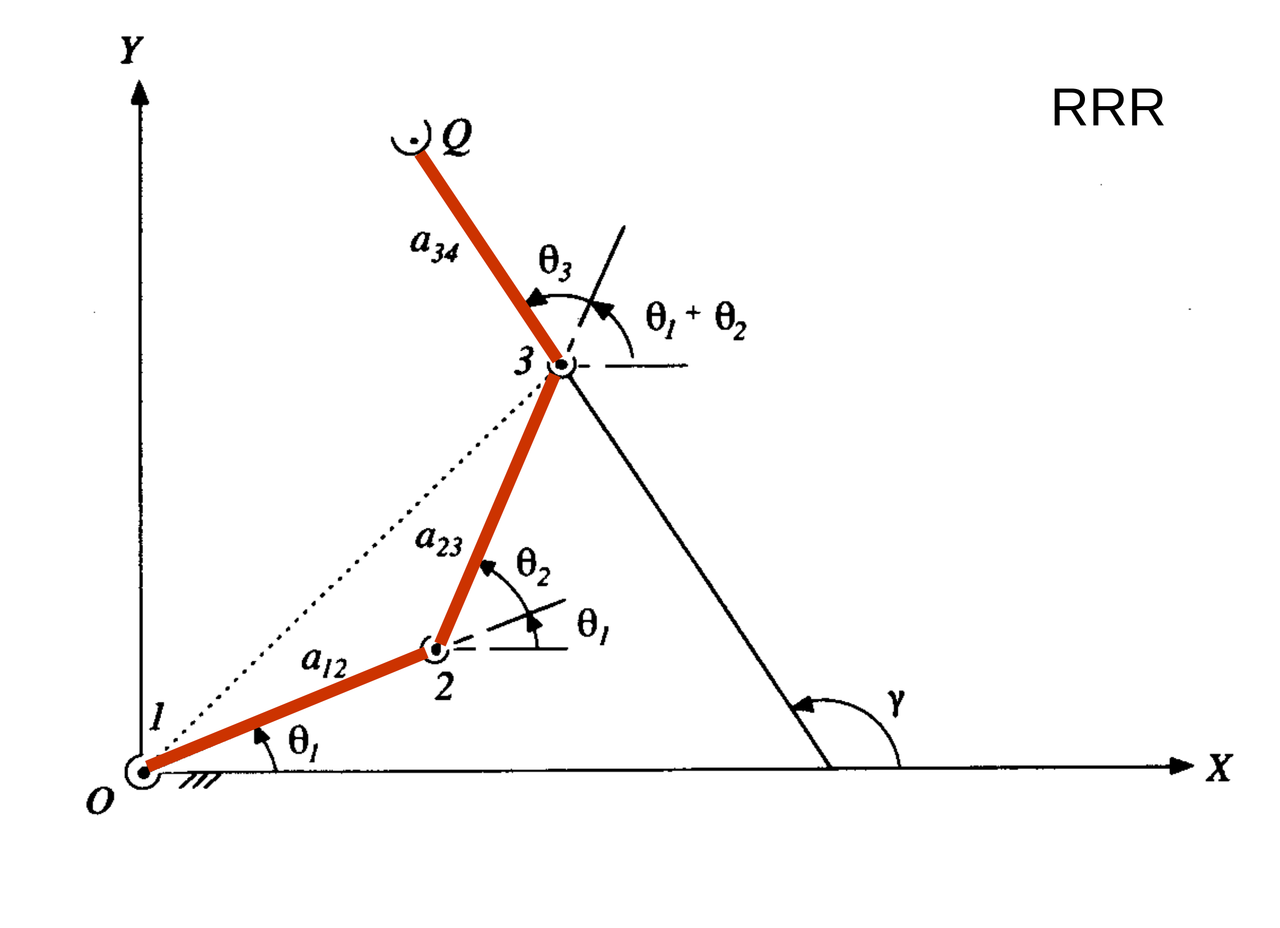
RRT



RTR



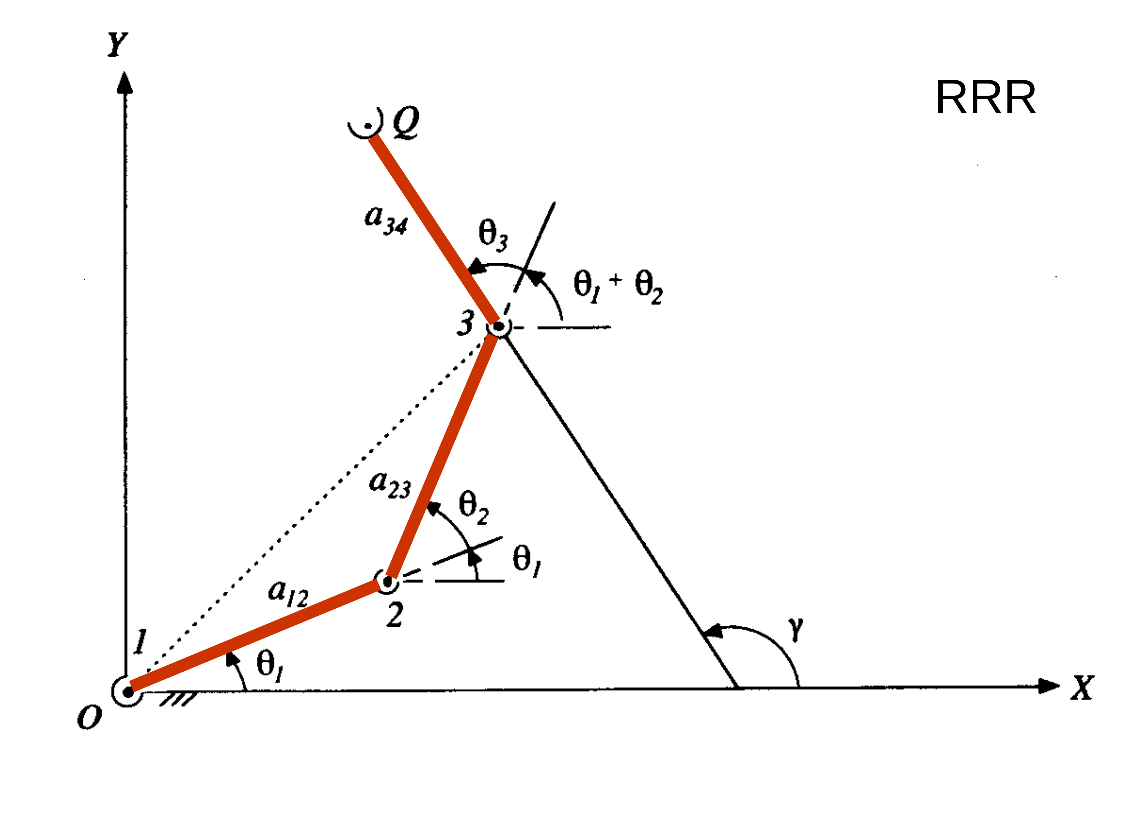
TRR



Direct kinematics: $\text{known}: \theta_1, \theta_2, \theta_3 \rightarrow \begin{cases} x_Q, y_Q - position \\ \gamma - orientation \end{cases}$

In general: DIRECT TASK

Computation of the pose, motion and forces at the end-effector of a robot arm from given actuator displacements, velocities, accelerations and forces



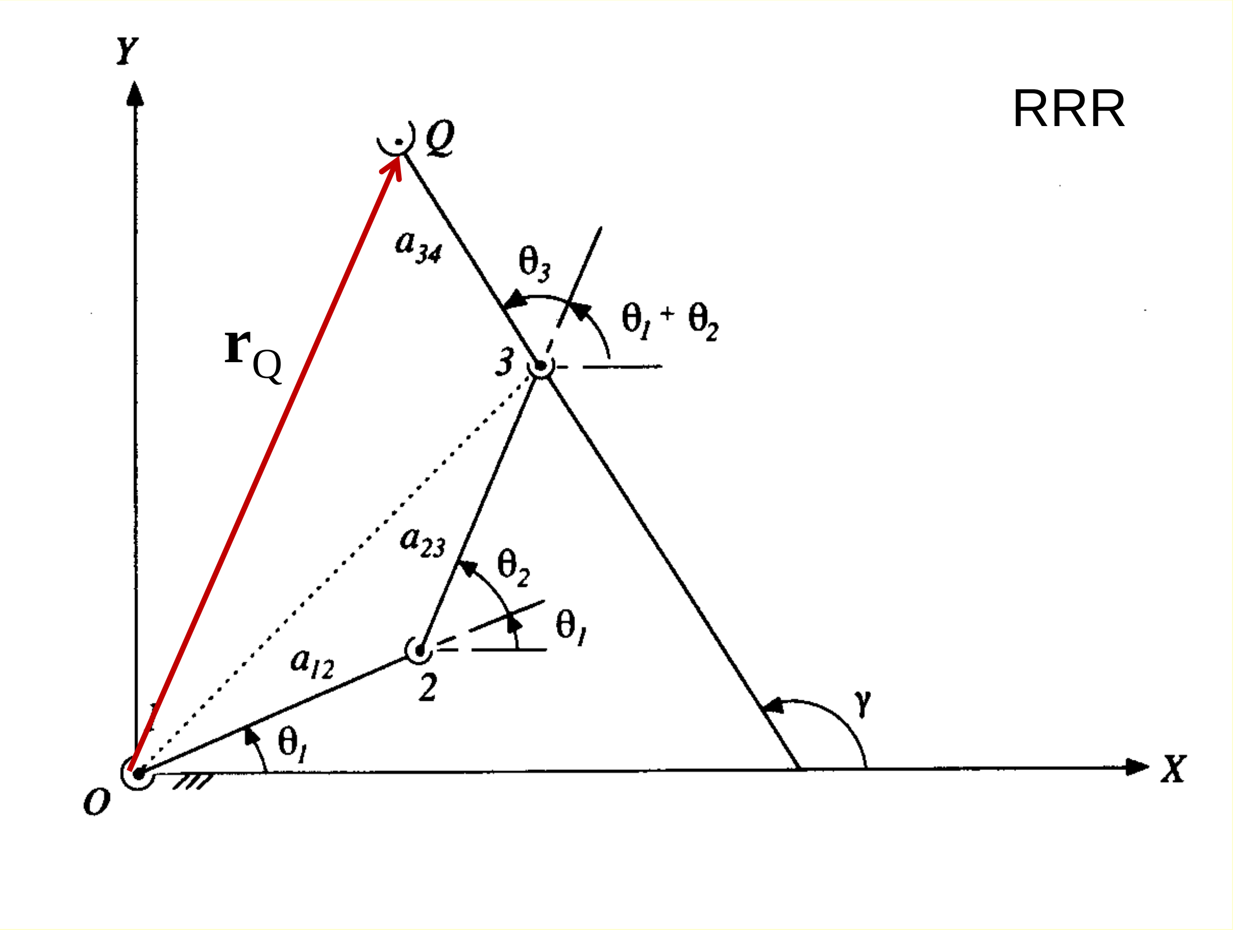
Inverse kinematics:

$$known:\left\{\begin{array}{c} x_Q, y_Q - position \\ \gamma - orientation \end{array}\right\} \rightarrow \theta_1, \theta_2, \theta_3$$

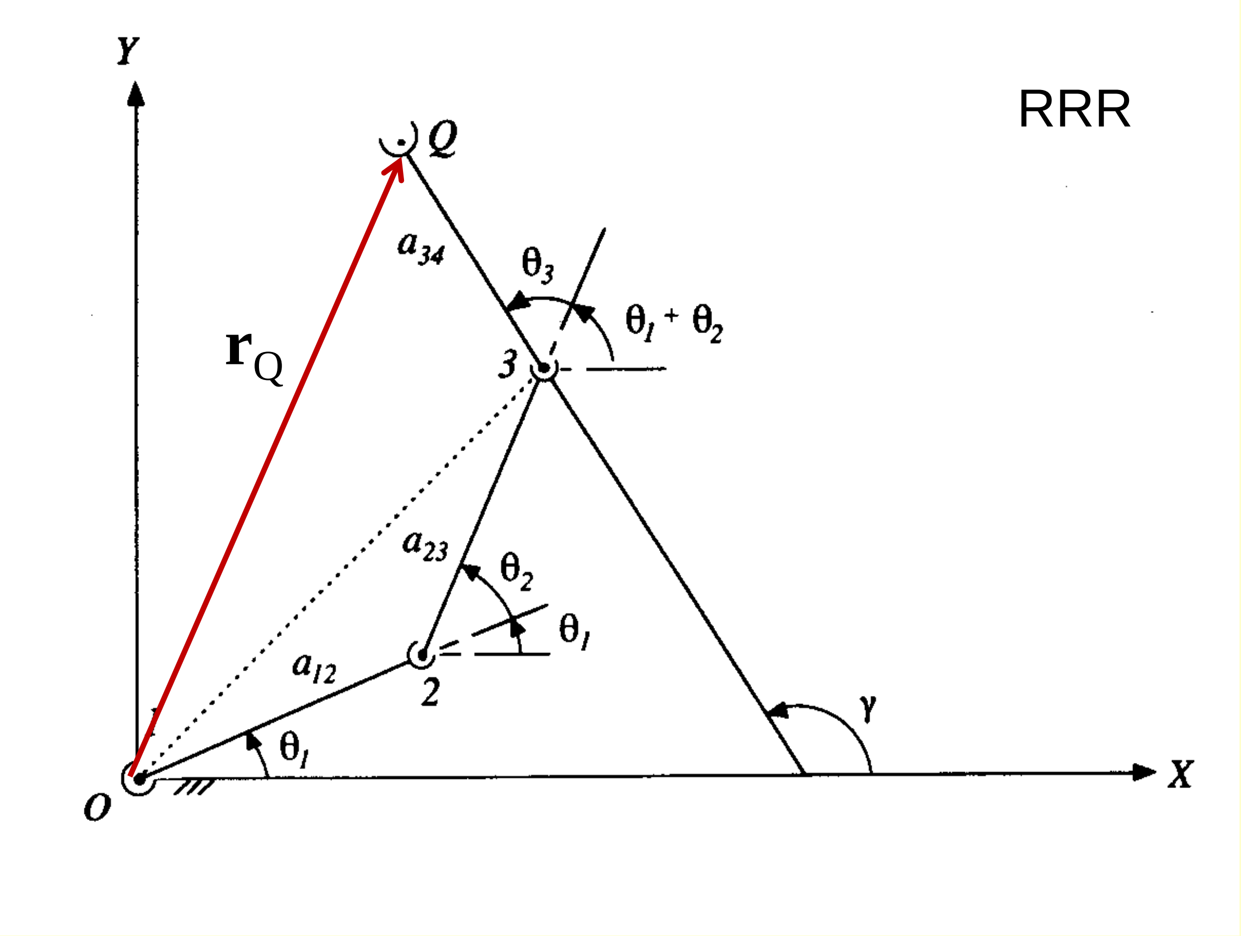
In general: INVERSE TASK

Computation of actuator forces, displacements, velocities and accelerations from given forces, pose and motion of the end-effector of a robot.

Direct kinematics using vector projections



$$\mathbf{r}_Q = \mathbf{a}_{12} + \mathbf{a}_{23} + \mathbf{a}_{34} \quad \leftarrow \quad \text{vectors}$$

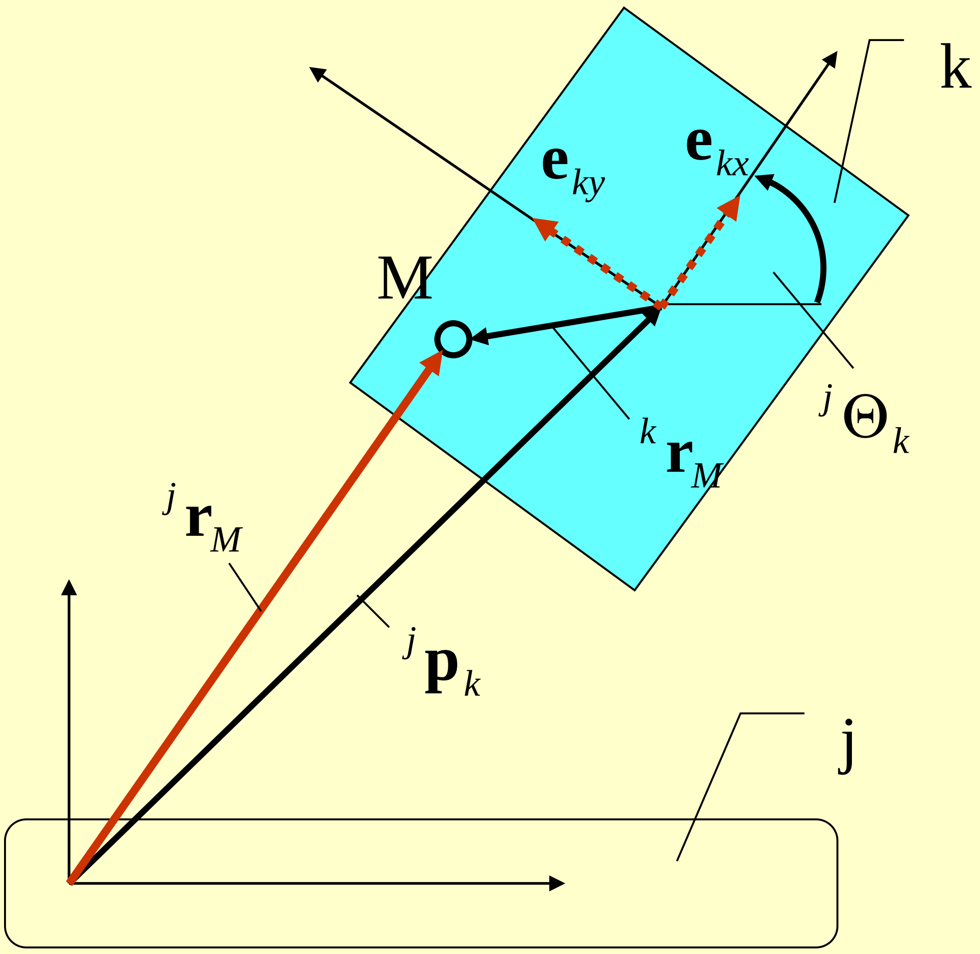


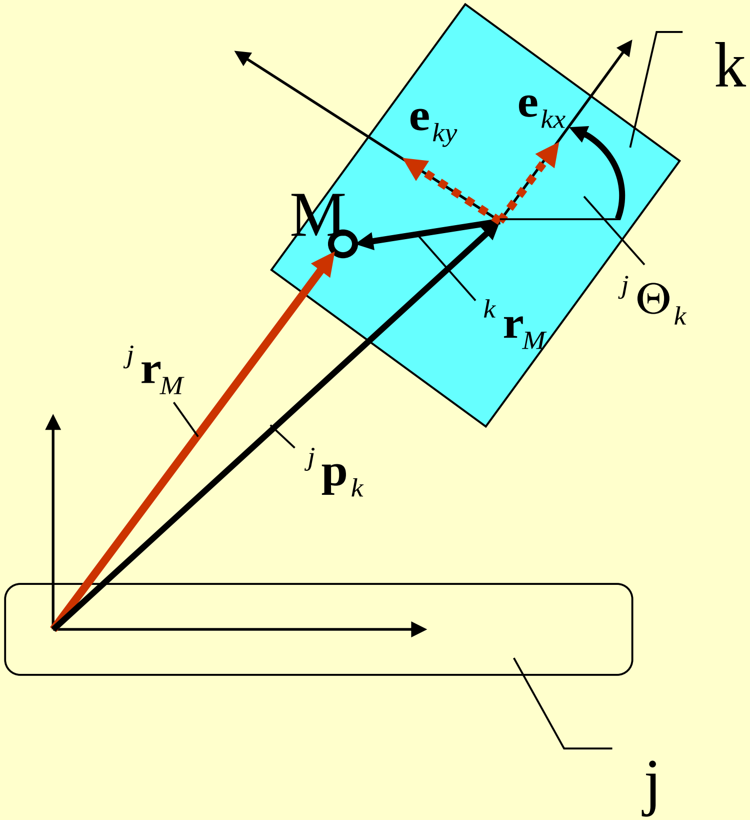
$$x_Q = a_{12}\cos\theta_1 + a_{23}\cos(\theta_1 + \theta_2) + a_{34}\cos(\theta_1 + \theta_2 + \theta_3)$$

$$y_Q = a_{12}\sin\theta_1 + a_{23}\sin(\theta_1 + \theta_2) + a_{34}\sin(\theta_1 + \theta_2 + \theta_3)$$

$$\gamma = \theta_1 + \theta_2 + \theta_3$$

Direct kinematics using Cartesian coordinates (absolute coordinates)





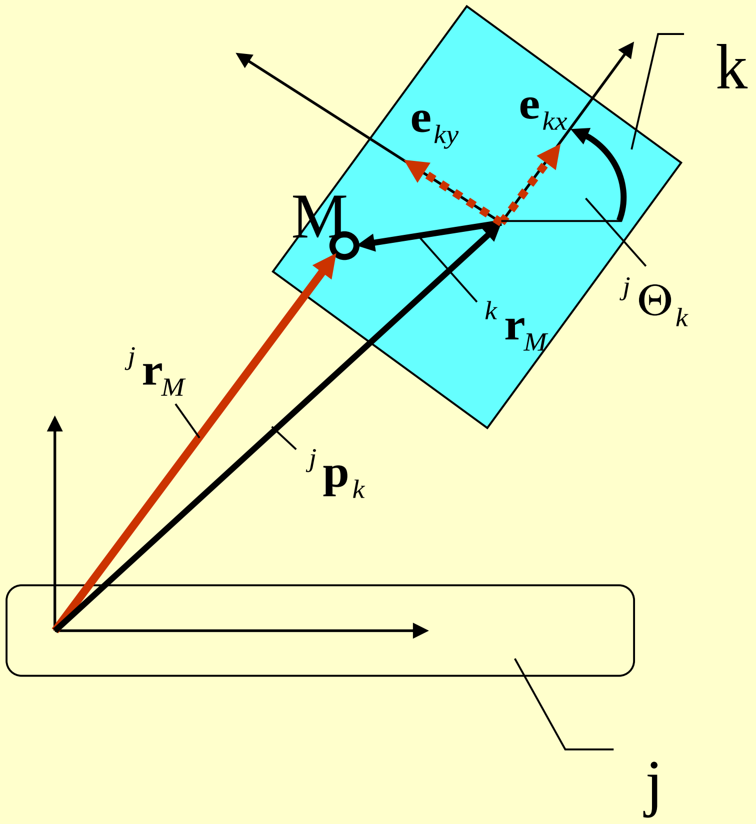
$$ {}^{j}\mathbf{r}_M = {}^{j}\mathbf{R}_k \; {}^{k}\mathbf{r}_M + {}^{j}\mathbf{p}_k $$

Rotation matrix

$$ {}^{j}\mathbf{R}_k = \left[{}^{j}\mathbf{e}_{kx} \quad {}^{j}\mathbf{e}_{ky} \right] = \begin{bmatrix} \cos {}^{j}\Theta_k & -\sin {}^{j}\Theta_k \\ \sin {}^{j}\Theta_k & \cos {}^{j}\Theta_k \end{bmatrix} $$

versors

$$ {}^{j}\mathbf{e}_{kx} = \begin{bmatrix} \cos {}^{j}\Theta_k \\ \sin {}^{j}\Theta_k \end{bmatrix} \qquad {}^{j}\mathbf{e}_{ky} = \begin{bmatrix} -\sin {}^{j}\Theta_k \\ \cos {}^{j}\Theta_k \end{bmatrix} $$



$$\begin{bmatrix} {}^{j}x_M \\ {}^{j}y_M \end{bmatrix} = \begin{bmatrix} \cos {}^{j}\Theta_k & -\sin {}^{j}\Theta_k \\ \sin {}^{j}\Theta_k & \cos {}^{j}\Theta_k \end{bmatrix} \begin{bmatrix} {}^{k}x_M \\ {}^{k}y_M \end{bmatrix} + \begin{bmatrix} {}^{j}x_k \\ {}^{j}y_k \end{bmatrix}$$

Properties of rotation matrix

Inversion=transposition!!!

$$ {}^{j}\mathbf{R}_{k}^{-1} = {}^{j}\mathbf{R}_{k}^{T} = \begin{bmatrix} \cos {}^{j}\Theta_{k} & \sin {}^{j}\Theta_{k} \\ -\sin {}^{j}\Theta_{k} & \cos {}^{j}\Theta_{k} \end{bmatrix} $$

$$ {}^{j}\mathbf{R}_{k}\,{}^{j}\mathbf{R}_{k}^{-1} = \begin{bmatrix} \cos {}^{j}\Theta_{k} & -\sin {}^{j}\Theta_{k} \\ \sin {}^{j}\Theta_{k} & \cos {}^{j}\Theta_{k} \end{bmatrix} \begin{bmatrix} \cos {}^{j}\Theta_{k} & \sin {}^{j}\Theta_{k} \\ -\sin {}^{j}\Theta_{k} & \cos {}^{j}\Theta_{k} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I} $$

identity matrix

Instead of

$$^{j}\mathbf{r}_M = {}^{j}\mathbf{R}_k \; {}^{k}\mathbf{r}_M + {}^{j}\mathbf{p}_k$$

we can use

$$^{j}\mathbf{r}_M = {}^{j}\mathbf{A}_k \, {}^{k}\mathbf{r}_M$$

Homogeneous transformation

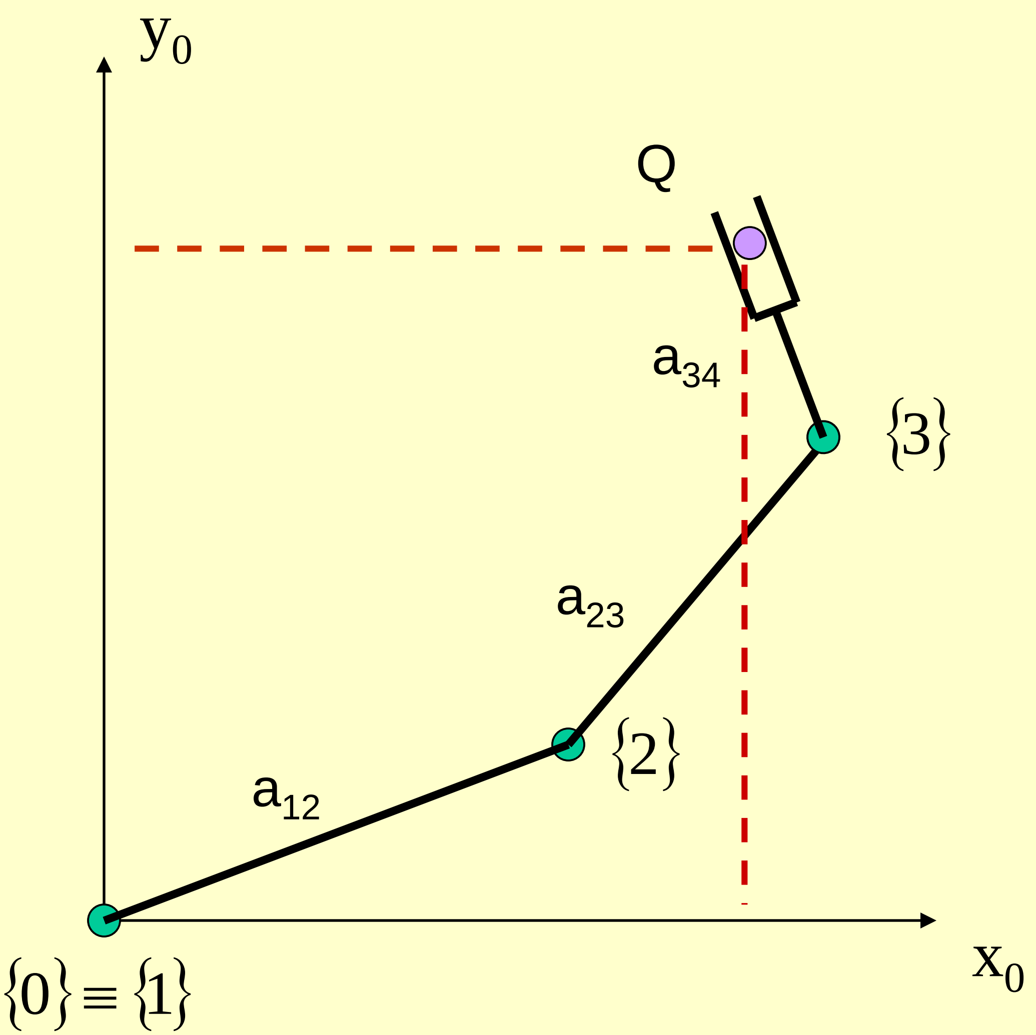
$$^{j}\mathbf{A}_k = \begin{bmatrix} {}^{j}\mathbf{R}_k & {}^{j}\mathbf{p}_\mathbf{k} \\ 0 \quad 0 & 1 \end{bmatrix}$$

$$^{j}\mathbf{A}_k = \begin{bmatrix} \cos {}^{j}\Theta_k & -\sin {}^{j}\Theta_k & {}^{j}x_k \\ \sin {}^{j}\Theta_k & \cos {}^{j}\Theta_k & {}^{j}y_k \\ 0 & 0 & 1 \end{bmatrix}$$

Vector of cart. coordinates

$$^{j}\mathbf{q}_k = \begin{bmatrix} {}^{j}\mathbf{p}_k^T & {}^{j}\Theta_k \end{bmatrix}^T = \begin{bmatrix} {}^{j}x_k & {}^{j}y_k & {}^{j}\Theta_k \end{bmatrix}^T$$

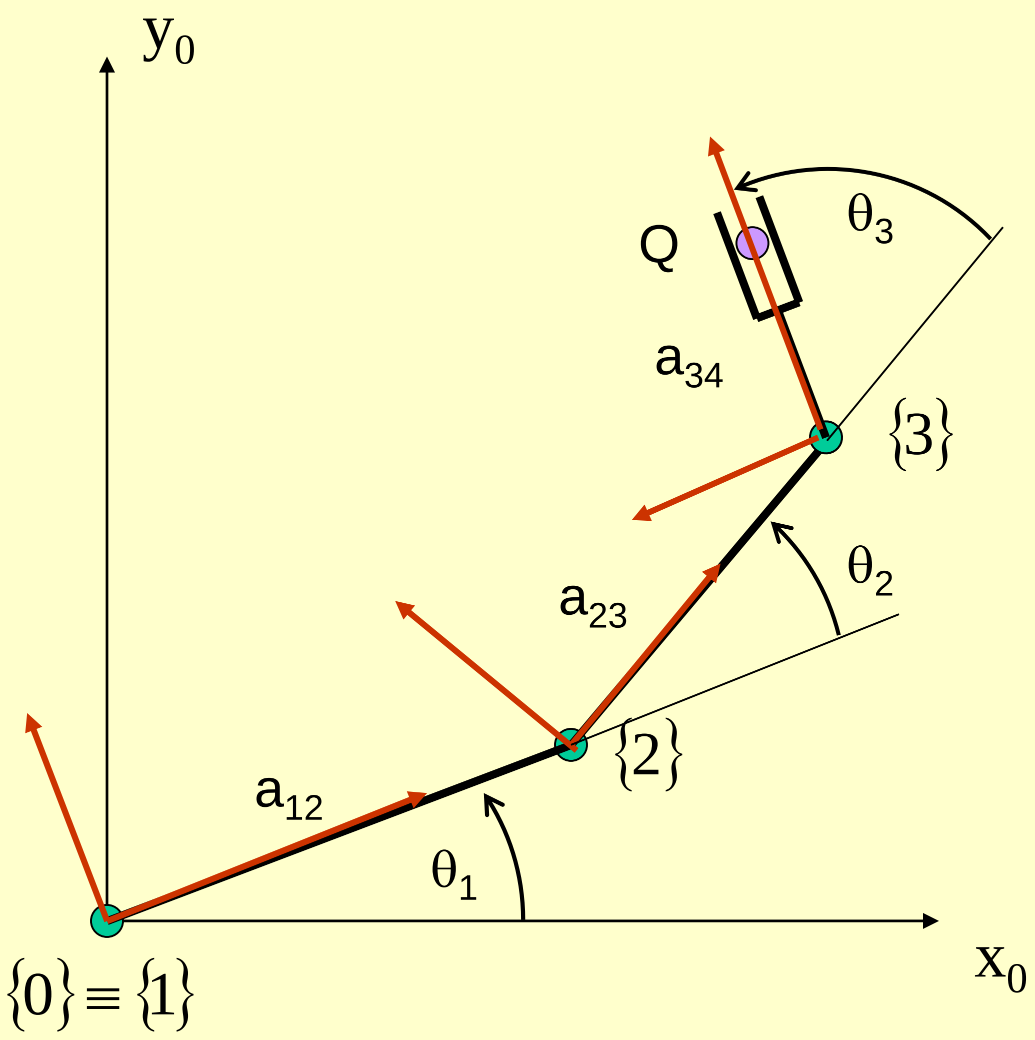
Example



Known:
point Q position on link 3
link dimensions
joint variables

Find:
point Q position in frame coordinate system $x_o y_o$

Example



1. Introduce link coordinate systems and joint variables

2. equations

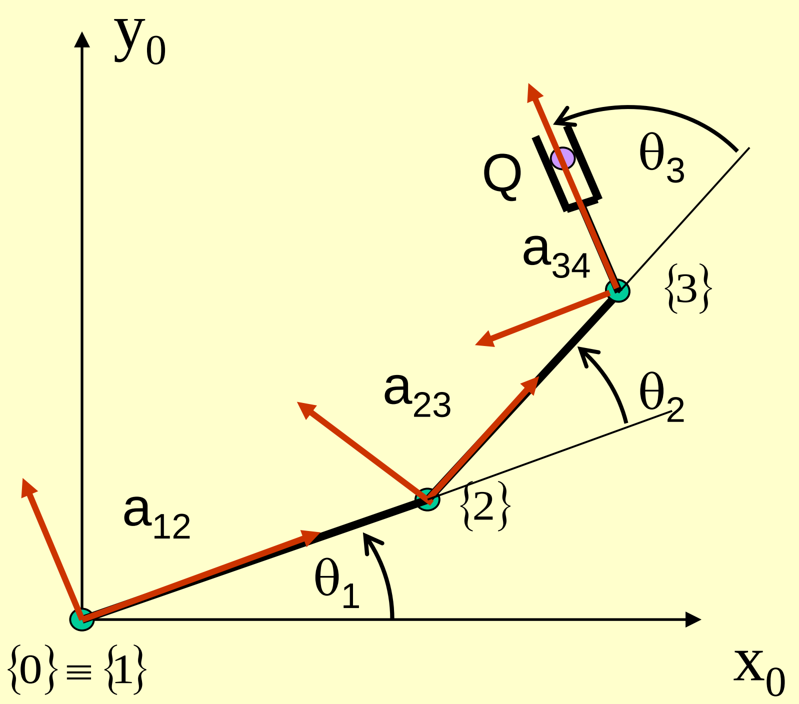
$$\begin{bmatrix} {}^3x_Q \\ {}^3y_Q \\ 1 \end{bmatrix} = \begin{bmatrix} a_{34} \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_Q \\ y_Q \\ 1 \end{bmatrix} = {}^0A_1 \cdot {}^1A_2 \cdot {}^2A_3 \begin{bmatrix} {}^3x_Q \\ {}^3y_Q \\ 1 \end{bmatrix}$$

$$ {}^{0}\mathbf{A}_1 = \begin{bmatrix} \cos\Theta_1 & -\sin\Theta_1 & 0 \\ \sin\Theta_1 & \cos\Theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}\mathbf{A}_2 = \begin{bmatrix} \cos\Theta_2 & -\sin\Theta_2 & a_{12} \\ \sin\Theta_2 & \cos\Theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

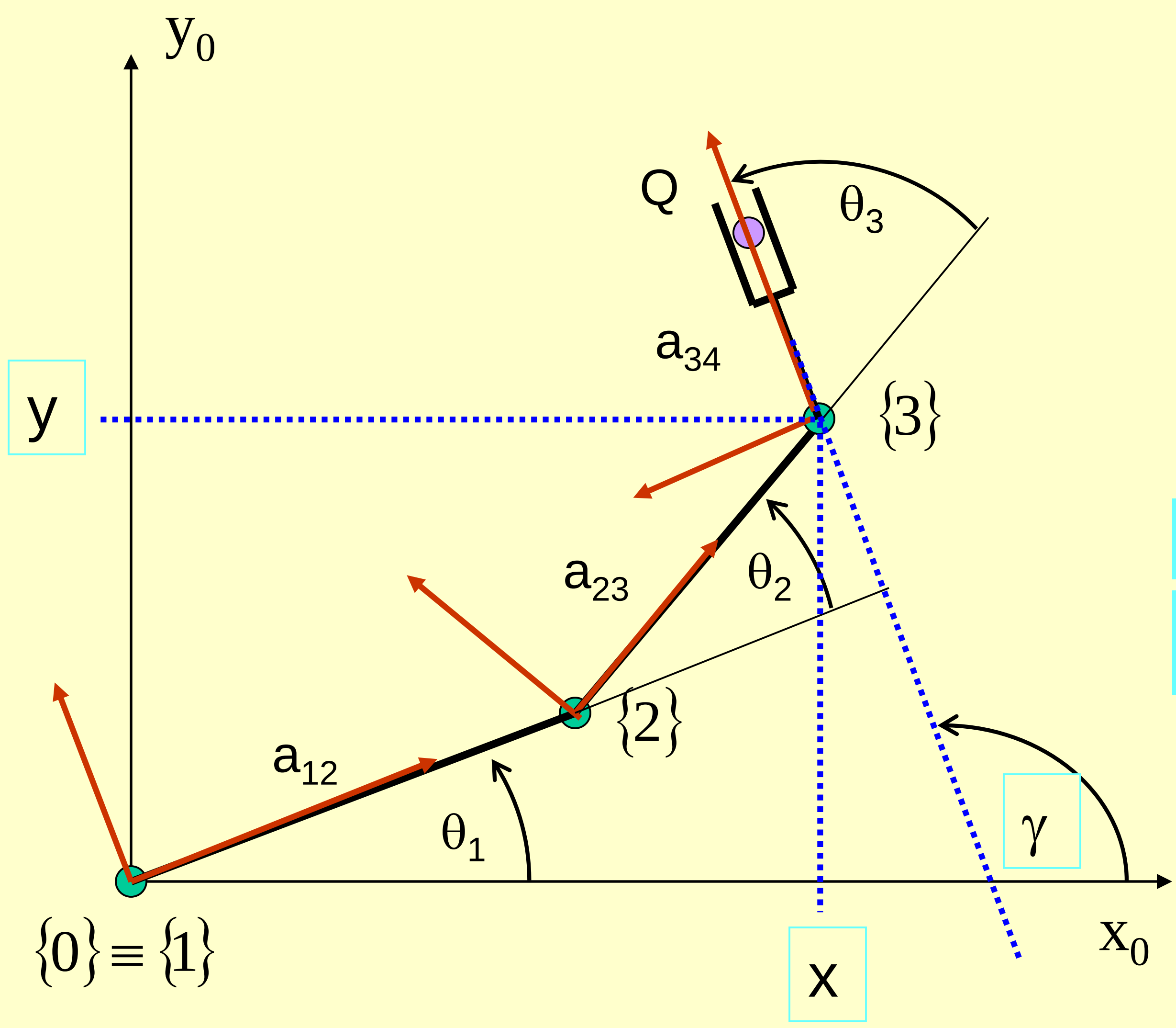
$$ {}^{2}\mathbf{A}_3 = \begin{bmatrix} \cos\Theta_3 & -\sin\Theta_3 & a_{23} \\ \sin\Theta_3 & \cos\Theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$



$$ \begin{bmatrix} x_Q \\ y_Q \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\Theta_1 & -\sin\Theta_1 & 0 \\ \sin\Theta_1 & \cos\Theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\Theta_2 & -\sin\Theta_2 & a_{12} \\ \sin\Theta_2 & \cos\Theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\Theta_3 & -\sin\Theta_3 & a_{23} \\ \sin\Theta_3 & \cos\Theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{3}x_Q \\ {}^{3}y_Q \\ 1 \end{bmatrix} $$

Manipulator 2D

Inverse kinematics

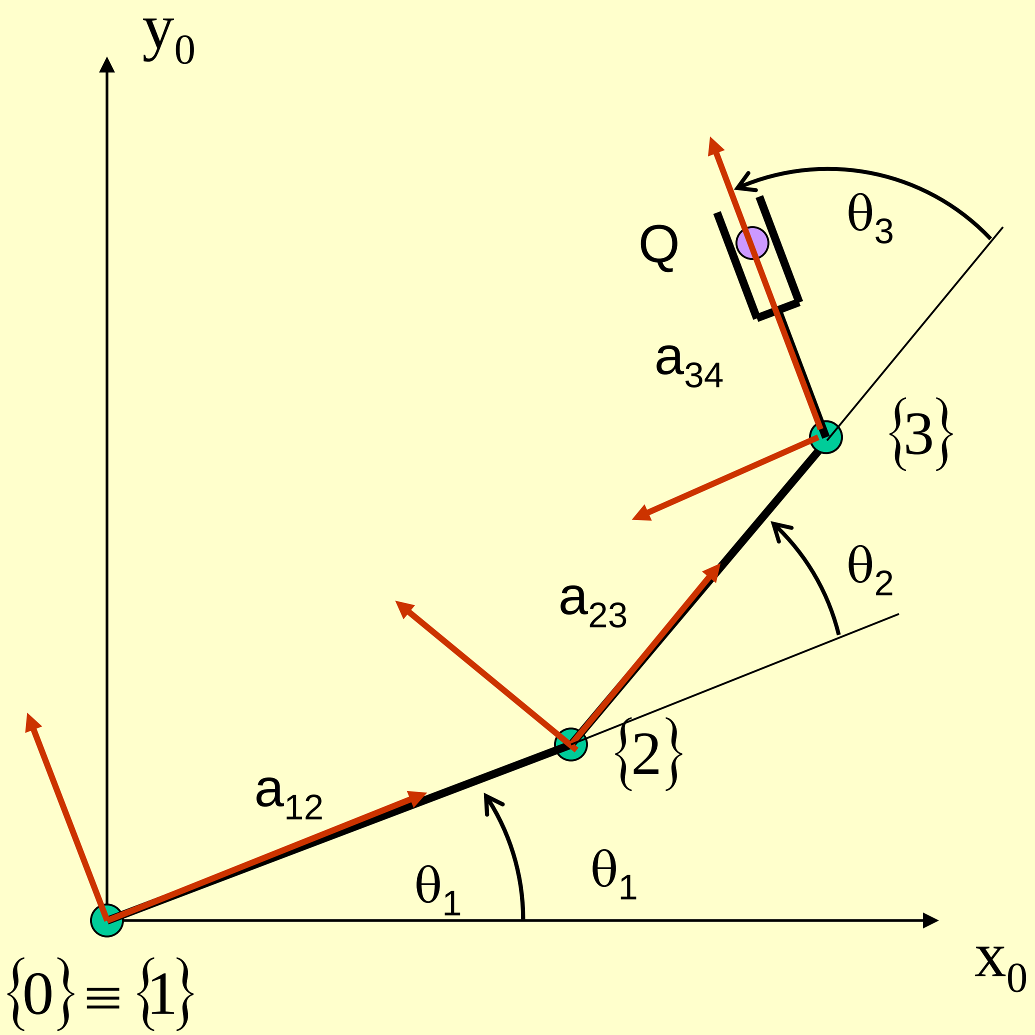


Given:

x, y, γ

Calculate:

$$\Theta_i = ?,\ i = 1, 2, 3$$



We already have:

$$\begin{bmatrix} x_Q \\ y_Q \\ 1 \end{bmatrix} = {}^0\mathbf{A}_1 \cdot {}^1\mathbf{A}_2 \cdot {}^2\mathbf{A}_3 \begin{bmatrix} {}^3x_Q \\ {}^3y_Q \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} {}^3x_Q \\ {}^3y_Q \\ 1 \end{bmatrix} = \begin{bmatrix} a_{34} \\ 0 \\ 1 \end{bmatrix}$$

$$
{}^{0}\mathbf{A}_1 = \begin{bmatrix} \cos\Theta_1 & -\sin\Theta_1 & 0 \\ \sin\Theta_1 & \cos\Theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

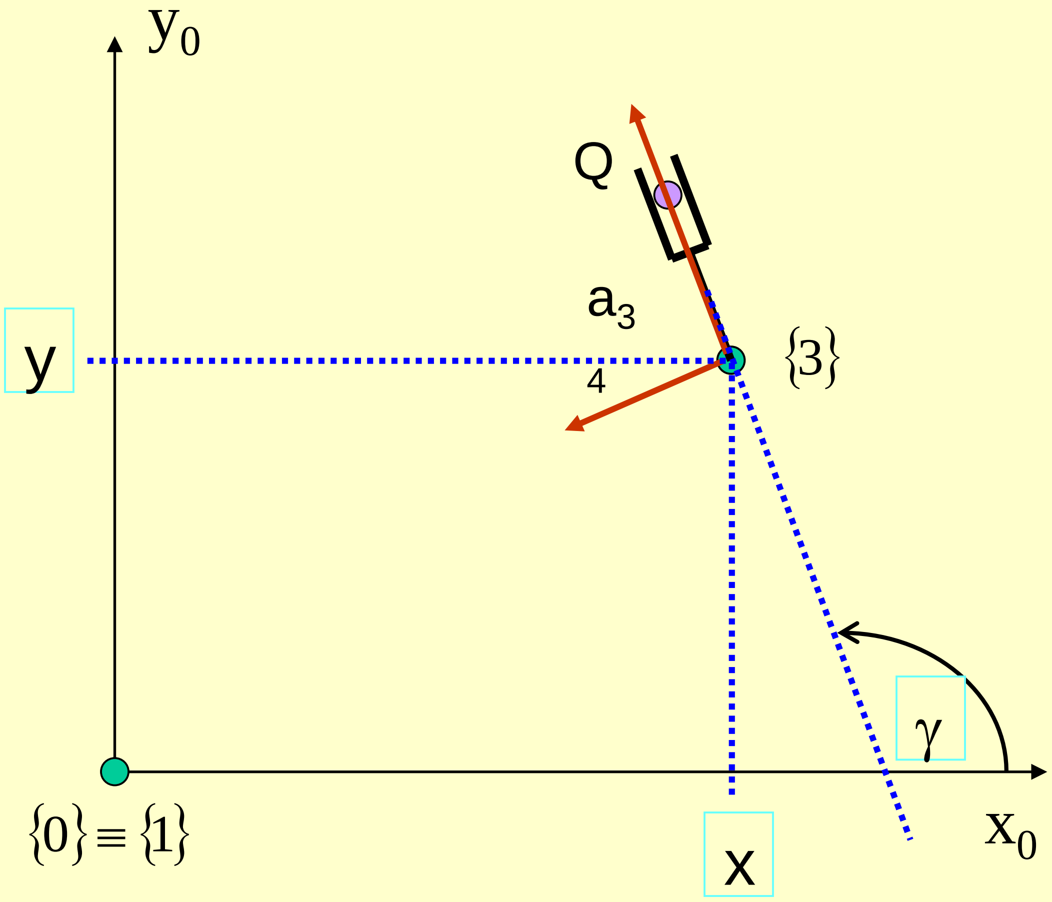
$$
{}^{1}\mathbf{A}_2 = \begin{bmatrix} \cos\Theta_2 & -\sin\Theta_2 & a_{12} \\ \sin\Theta_2 & \cos\Theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{2}\mathbf{A}_3 = \begin{bmatrix} \cos\Theta_3 & -\sin\Theta_3 & a_{23} \\ \sin\Theta_3 & \cos\Theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$^{0}\mathbf{A}_1 \cdot {}^{1}\mathbf{A}_2 \cdot {}^{2}\mathbf{A}_3 = {}^{0}\mathbf{A}_3$$

$$^{0}\mathbf{A}_3 =$$

$$\begin{bmatrix} \cos(\Theta_1+\Theta_2+\Theta_3) & -\sin(\Theta_1+\Theta_2+\Theta_3) & a_{12}\cos\Theta_1 + a_{23}\cos(\Theta_1+\Theta_2) \\ \sin(\Theta_1+\Theta_2+\Theta_3) & \cos(\Theta_1+\Theta_2+\Theta_3) & a_{12}\sin\Theta_1 + a_{23}\sin(\Theta_1+\Theta_2) \\ 0 & 0 & 1 \end{bmatrix}$$



$$
{}^{0}\mathbf{T}_3 = \begin{bmatrix} \cos\gamma & -\sin\gamma & x \\ \sin\gamma & \cos\gamma & y \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{0}\mathbf{A}_3 = \begin{bmatrix} \cos(\Theta_1+\Theta_2+\Theta_3) & -\sin(\Theta_1+\Theta_2+\Theta_3) & a_{12}\cos\Theta_1 + a_{23}\cos(\Theta_1+\Theta_2) \\ \sin(\Theta_1+\Theta_2+\Theta_3) & \cos(\Theta_1+\Theta_2+\Theta_3) & a_{12}\sin\Theta_1 + a_{23}\sin(\Theta_1+\Theta_2) \\ 0 & 0 & 1 \end{bmatrix}
$$

Comparison of matrices:

$$
{}^{0}\mathbf{A}_3 = {}^{0}\mathbf{T}_3 = \begin{bmatrix} \cos\gamma & -\sin\gamma & x \\ \sin\gamma & \cos\gamma & y \\ 0 & 0 & 1 \end{bmatrix}
$$

It gives 4 equations

$$\begin{cases} \cos\gamma = \cos(\Theta_1 + \Theta_2 + \Theta_3) & (a) \\ \sin\gamma = \sin(\Theta_1 + \Theta_2 + \Theta_3) & (b) \\ x = a_{12}\cos\Theta_1 + a_{23}\cos(\Theta_1 + \Theta_2) & (c) \\ y = a_{12}\sin\Theta_1 + a_{23}\sin(\Theta_1 + \Theta_2) & (d) \end{cases}$$

$$\Theta_1 = \ldots \qquad \Theta_2 = \ldots \qquad \Theta_3 = \ldots$$

$$\begin{cases} \cos\gamma = \cos(\Theta_1 + \Theta_2 + \Theta_3) & (a) \\ \sin\gamma = \sin(\Theta_1 + \Theta_2 + \Theta_3) & (b) \\ x = a_{12}\cos\Theta_1 + a_{23}\cos(\Theta_1 + \Theta_2) & (c) \\ y = a_{12}\sin\Theta_1 + a_{23}\sin(\Theta_1 + \Theta_2) & (d) \end{cases}$$

- (c)^2 and (d)^2 sum gives:

$$x^2 + y^2 = a_{12}^2 + a_{23}^2 + 2a_{12}a_{23}u$$

where:

$$u = \cos\Theta_1 \cos(\Theta_1 + \Theta_2) + \sin\Theta_1 \sin(\Theta_1 + \Theta_2)$$

$$u = \cos\Theta_1 \cos(\Theta_1 + \Theta_2) + \sin\Theta_1 \sin(\Theta_1 + \Theta_2)$$

Using:

$$\cos(\Theta_1 + \Theta_2) = \cos\Theta_1 \cos\Theta_2 - \sin\Theta_1 \sin\Theta_2$$
$$\sin(\Theta_1 + \Theta_2) = \sin\Theta_1 \cos\Theta_2 + \cos\Theta_1 \sin\Theta_2$$

$$u = \cos\Theta_1 \cos(\Theta_1 + \Theta_2) + \sin\Theta_1 \sin(\Theta_1 + \Theta_2)$$

One can obtain

$$u = \cos\Theta_1 \cos\Theta_1 \cos\Theta_2 - \cos\Theta_1 \sin\Theta_1 \sin\Theta_2 +$$
$$+ \sin\Theta_1 \sin\Theta_1 \cos\Theta_2 + \sin\Theta_1 \cos\Theta_1 \sin\Theta_2 =$$
$$= \left(\cos^2\Theta_1 + \sin^2\Theta_1\right)\cos\Theta_2 = \cos\Theta_2$$

$$u = \cos\Theta_2$$

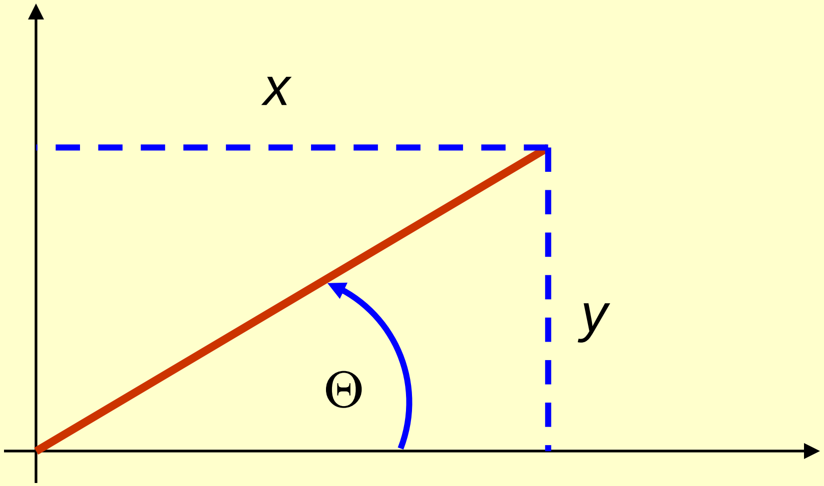
$$x^2 + y^2 = a_{12}^2 + a_{23}^2 + 2a_{12}a_{23}u$$

$$x^2 + y^2 = a_{12}^2 + a_{23}^2 + 2a_{12}a_{23}\cos\Theta_2$$

$$\cos\Theta_2 = \frac{x^2 + y^2 - a_{12}^2 - a_{23}^2}{2a_{12}a_{23}}$$

$$\Theta_2 = \operatorname{atan2}\left(\pm\sqrt{1-\cos^2\Theta_2}, \cos\Theta_2\right)$$

Function *atan2(y,x)* – *(1)*



atan2(2,2) = pi/4

atan2(-2,-2)=-3pi/4

but

arctan(2/2) =

arctan[(-2)/(-2)]=pi/4

$-\pi <= atan2(y,x) <= \pi$

atan2(y,x) calculates arctan(y/x) for any y, x

$$\begin{cases} \sin\Theta = a \\ \cos\Theta = b \end{cases} \rightarrow \Theta = \mathrm{atan2}(a, b)$$

$$a\cos\Theta + b\sin\Theta = 0$$

$$\downarrow$$

1. $\Theta = \mathrm{atan2}(\mathrm{a}, -\mathrm{b})$
2. $\Theta = \mathrm{atan2}(-\mathrm{a}, \mathrm{b})$

$$\left.\begin{aligned} a\cos\Theta - b\sin\Theta &= c \\ a\sin\Theta + b\cos\Theta &= d \end{aligned}\right\} \rightarrow$$

$$\Theta = \mathrm{atan2}\left(\mathrm{ad} - \mathrm{bc}, \mathrm{ac} - \mathrm{bd}\right)$$

Two configurations

$$\Theta_2 = \operatorname{atan2}\left(\pm\sqrt{1-\cos^2\Theta_2}\,,\cos\Theta_2\right)$$